Ma2a Practical – Recitation 9

December 6, 2024

Exercise 1. Study the singularities of the following equations, and determine the exponent of regular singularities. Discuss the existence of a basis of Frobenius solutions.

1. Euler equation: $x^2y'' + xy' + y = 0$
2. Legendre equation: $(1-x^2)y'' - 2xy' + \alpha(\alpha+1)y = 0$, where $\alpha \in \mathbb{R}$.
3. Bessel equation: $x^2y'' + xy' + (x^2 - \nu^2)y = 0$, where $\nu \in \mathbb{R}$.
4. Laguerre equation: $xy'' + (\alpha + 1 - x)y' + ny = 0$, where $\alpha \in \mathbb{R}$ and $n \in \mathbb{N}_{>0}$.
5. $(x-1)^3(x+2)y'' + 2xy' + 3(x-2)y = 0$.

laplace transform

Exercise 2. (Chapter 6.4 Exercise 1; Laplace transform) Use Laplace transform to solve the IVP:

$$y'' + y = f(t); \quad y(0) = 0, \quad y'(0) = 1; \quad f(t) = \begin{cases} 1, & 0 \leqslant t < 3\pi \\ 0, & 3\pi \leqslant t < \infty \end{cases}$$

Exercise 3. (Riccati equation) Consider the general Riccati equation

$$y' + p(x)y = g(x)y^2 + h(x). \tag{1}$$

1. Conisder the change of variable $y(x) = -\frac{u'(x)}{g(x)u(x)}$. Compute $y(x)$ in terms of $u(x)$ and $u'(x)$.
2. Show that the original ODE is $u''(x) + a(x)u'(x) + b(x)u(x) = 0$ where $a(x) = p(x) - g'(x)g(x), b(x) = g(x)h(x)$.
3. Conversely, show that any second order linear homogenous ODE of form $u'' + a(x)u' + b(x)u = 0$ can be transformed into a first order ODE (Riccati equation).

solve difference equation

Solution

1. *Legendre equation.* There are singularities when $1-x^2=0$, i.e. at $x=\pm 1$.

   - *Singularity at* $x=-1$. We have

$$-(x+1)\frac{2x}{1-x^2}=\frac{2x}{(x-1)}\underset{x\to-1}{\longrightarrow}1.$$

$$(x+1)^2\frac{\alpha(\alpha+1)}{1-x^2}=-(x+1)\frac{\alpha(\alpha+1)}{x-1}\underset{x\to-1}{\longrightarrow}0.$$

     We deduce that $x=-1$ is a regular singularity, with indicial equation

$$r(r-1)+r=0\Leftrightarrow r(r-2)=0.$$

     The exponents of the singularity are $r_1=r_2=0$.

   - *Singularity at* $x=1$. We have

$$-(x-1)\frac{2x}{1-x^2}=\frac{2x}{(x+1)}\underset{x\to 1}{\longrightarrow}1.$$

$$(x-1)^2\frac{\alpha(\alpha+1)}{1-x^2}=-(x-1)\frac{\alpha(\alpha+1)}{x+1}\underset{x\to 1}{\longrightarrow}0.$$

     We deduce that $x=1$ is a regular singularity, with indicial equation

$$r(r-1)+r=0\Leftrightarrow r(r-2)=0.$$

     The exponents are again $r_1=r_2=0$.

2. *Bessel equation.* The only singularity is $x=0$. We have

$$x\frac{x}{x^2}=1\underset{x\to 0}{\longrightarrow}1,$$

$$x^2\frac{x^2-\nu^2}{x^2}=x^2-\nu^2\underset{x\to 0}{\longrightarrow}-\nu^2.$$

   So $x=0$ is a regular singularity and the indical equation is

$$r(r-1)+r-\nu^2=0.$$

   So the exponents are $r_\pm=\pm|\nu|$. The roots r_+ gives rise to a Frobenius solution. The root r_- gives a Frobenius solution provided that $r_+-r_-=2|\nu|\notin\mathbb{N}$.

3. *Laguerre equation.* There is a singularity at $x=0$. We have

$$x\frac{\alpha+1-x}{x}=\alpha+1-x\underset{x\to 0}{\longrightarrow}\alpha+1,$$

$$x^2\frac{n}{x} = xn \underset{x\to 0}{\longrightarrow} 0.$$

We conclude that $x = 0$ is a regular singularity, with indicial equation

$$r(r-1) + (\alpha+1)r = 0 \Leftrightarrow r(r+\alpha) = 0.$$

The exponents are $r_1 = 0$ and $r_2 = -\alpha$. If $\alpha \geqslant 0$, then r_1 gives a Frobenius solution and r_2 as well provided that $\alpha \notin \mathbb{N}$. If $\alpha < 0$, the r_2 gives a Frobenius solution, and r_1 as well provided that $-\alpha \notin \mathbb{N}$.

In particular, when $\alpha \notin \mathbb{Z}$ there are always two Frobenius solutions.

4. There are singularities at $x = \pm 2$.

   - *Singularity at* $x = -2$. We have

$$(x+2)\frac{2x}{(x-2)^2(x+2)} = \frac{2x}{(x-2)^2} \underset{x\to -2}{\longrightarrow} -\frac{1}{4},$$

$$(x+2)^2\frac{3(x-2)}{(x-2)^2(x+2)} = \frac{3(x+2)}{(x-2)} \underset{x\to -2}{\longrightarrow} 0.$$

     So $x = -2$ is a regular singularity, with indicial equation

$$r(r-1) - \frac{r}{4} = 0 \Leftrightarrow r(r-\frac{5}{4}) = 0.$$

     The exponents are $r_1 = 0$ and $r_2 = \frac{5}{4}$, they both give rise to Frobenius solutions since $r_2 - r_1 \notin \mathbb{N}$.

   - *Singularity at* $x = 1$. We have

$$(x-1)\frac{2x}{(x-1)^3(x+2)} = \frac{2x}{(x-1)^2(x+2)} \underset{x\to 1^\pm}{\longrightarrow} \pm\infty,$$

     so the singularity $x = 1$ is irregular.

Solution

The idea to solve the problem is doing the Laplace transform on the both side of the equation. We write down the Laplace transform $\Psi(s) = \mathcal{L}\{y(t)\}$, then the Laplace transform of y'' is given as $s^2 * \Psi(s) - sy(0) - y'(0) = s^2 * \Psi(s) - 1$. Besides, $f(t) = 1 - u_{3\pi}(t)$ where $u_{3\pi}(t)$ is the step function. By searching the Laplace transform table, the Laplace transform of 1 is $1/s$ and the Laplace transform of $u_{3\pi}(t)$ is $e^{-3\pi s}/s$. Thus, $\mathcal{L}\{f(t)\} = \frac{1-e^{-3\pi s}}{s}$ (one could also calculate the Laplace transform of $f(t)$ directly, it's not hard). Sum the Laplace transform together, we have $(s^2+1) * \Psi(s) - 1 = \frac{1-e^{-3\pi s}}{s}$, equivalently,

$$\Psi(s) = \frac{1}{s^2+1}(1 + \frac{1}{s} - \frac{e^{-3\pi s}}{s})$$

$$= \frac{1}{s^2+1} + \frac{1}{s} - \frac{s}{s^2+1} + e^{-3\pi s}(\frac{s}{s^2+1} - \frac{1}{s}).$$

Finally, we calculate the Laplace inverse transform of $\Psi(s)$ with the table. The inverse Laplace transform of $\frac{1}{s^2+1}$ is $\sin t$. Besides, the inverse Laplace transform of $\frac{1}{s}$ is 1, and the inverse Laplace transform of $\frac{s}{s^2+1}$ is $\cos t$. Using the fact that the Laplace transform of $u_c(t)f(t-c)$ is $e^{-cs}F(s)$ where $F(s)$ is the Laplace transform of $f(t)$, the inverse Laplace transform of $\Psi(s)$ is $sint + 1 - cost + u_{3\pi}(t)(cos(t-3\pi) - 1) = sint + 1 - cost - u_{3\pi}(t)(cost + 1)$. It is exactly the solution for this IVP.

Solution

1. We start by expressing y' in terms of u and its derivatives. We have

$$\begin{aligned} y' &= \frac{-u''gu + u'\left(g'u + gu'\right)}{g^2u^2} \\ &= -\frac{u''}{gu} + \frac{u'g'}{g^2u} + \frac{(u')^2}{gu^2}. \end{aligned}$$

We thus have

$$\begin{aligned} y' + py - gy^2 &= -\frac{u''}{gu} + \frac{u'g'}{g^2u} + \frac{(u')^2}{gu^2} - \frac{pu'}{gu} - \frac{(u')^2}{gu^2} \\ &= \frac{1}{gu}\left(-u'' + u'\left(\frac{g'}{g} - p\right)\right) \end{aligned}$$

At this stage, we plug this into Equation (1) and multiply everything by gu, and obtain the homogeneous ODE

$$u'' + u'\left(p - \frac{g'}{g}\right) + (gh)u = 0.$$

2. Conversely, given an ODE $u'' + a(x)u' + b(x)u = 0$, the function $y(x) = -\frac{u'}{u}$ satisfies the Riccati equation

$$y' + a(x)y = y^2 + b(x).$$